CS 598: Communication Cost Analysis of Algorithms
Lecture 5: memory- and communication-efficient LU factorization

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Segmented scan

Given a $n \times P$ matrix A, compute $n \times P$ matrix $B = S(A)$, where

$$B(i,j) = \sum_{k=1}^{j} A(i,k)$$

$$A_{\text{odd}} = [A(:,1), A(:,3), \ldots A(:,P-1)],$$
$$A_{\text{even}} = [A(:,2), A(:,4), \ldots A(:,P)].$$

Now, observe that $B_{\text{even}} = S(A_{\text{odd}} + A_{\text{even}})$ and that $B_{\text{odd}} = B_{\text{even}} - A_{\text{even}}$.

The above version is a 'postfix' sum, a 'prefix' sum $B = R(A)$ is more standard

$$B(i,j) = \sum_{k=1}^{j-1} A(i,k)$$

Now, $B_{\text{even}} = R(A_{\text{odd}} + A_{\text{even}})$ and $B_{\text{odd}} = B_{\text{even}} + A_{\text{even}}$. Neither version requires an additive inverse. A *scan* is a prefix sum with an arbitrary $+$ operator.

Parallel segmented scan

The parallel prefix sum is the first parallel algorithm many people learn

$$T_{\text{scan}}(P) = T_{\text{scan}}(P/2) + 2 = 2\log_2(P)$$

for $T \in \{\text{computation, communication, synchronization}\}$.
So we can trivially get

$$T_{\text{seg-scan}}(n, P) = T_{\text{seg-scan}}(n, P/2) + 2 \cdot \alpha + 2n \cdot \beta = 2\log_2(P) \cdot \alpha + 2n\log_2(P) \cdot \beta$$

MPI::Scan does the trivial algorithm :(

Note 1: the n scans are *independent*
Note 2: parallel scan discards half the processors at each step

Butterfly Idea: assign $n/2$ of the scans to the other half of the processors

$$T_{\text{seg-scan}}(n, P) = T_{\text{seg-scan}}(n/2, P/2) + 2 \cdot \alpha + (n/2) \cdot \beta = 2\log_2(P) \cdot \alpha + n \cdot \beta$$

BSP Idea: transpose A and have each processor compute n/P scans sequentially

Senders vs receivers in a wrapped butterfly

We proved in lecture that the senders in the wrapped butterfly (Träff and Ripke) algorithm are independent

- I thought the showing this for receivers would require some work
- some students were more clever than me...
- the set of receivers at the next level is the set of senders in the previous with a flipped bit
- if $x \neq y$, flipping the same bit preserves the inequality
  - if we flip a bit that is different in x and y, the bits remain different
- HW 1 take-away: *simplicity is attained by finding the right perspective*

Homework 2

- problem 1 is Strassen's algorithm
    - recursion dragon is back
    - algorithms are given, your task: analysis
    - should be analogous to recursive MM and LU
- problem 2 is radix sort
    - algorithm given, last part requires minor modification
    - your primary task is again cost analysis
    - uses HW 1 problem 1!
- if you did not complete HW 1, remember the lowest homework grade is disregarded, but not the second lowest...

Recursive LU factorization: analysis

LU requires two recursive calls and $O(1)$ matrix multiplications

$$T_{\text{LU}}(n, P) = 2T_{\text{LU}}(n/2, P) + O\Big(\log(P) \cdot \alpha + \frac{n^2}{P^{2/3}} \cdot \beta\Big)$$

the bandwidth cost decreases geometrically (by a factor of 2) at each level.
If we allgather the matrix at the base cases, each has a cost of

$$T_{\text{LU}}(n_0, P) = O(\log(P) \cdot \alpha + n_0^2 \cdot \beta)$$

Q: What choice of n_0 makes the base cases have bandwidth cost less than $\frac{n^2}{P^{2/3}}$?

$$T_{\text{bc}}(n, n_0, P) = \frac{n}{n_0} T_{\text{LU}}(n_0, P)$$

A: we would want select is $n_0 = n/P^{2/3}$, giving a total cost of

$$T_{\text{LU}}(n, P) = O(P^{2/3} \cdot \log(P) \cdot \alpha + \frac{n^2}{P^{2/3}} \cdot \beta\Big)$$

In the BSP model, we lose the $\log(P)$ factors in synchronization cost.

Recursive triangular inversion: analysis

The two recursive calls within triangular inversion are independent, so we can perform them simultaneously with half of the processors

$$\begin{aligned} T_{\text{Tri-Inv}}(n,P) &= T_{\text{Tri-Inv}}(n/2,P/2) + O(T_{\text{MM}}(n,P)) \\ &= T_{\text{Tri-Inv}}(n/2,P/2) + O\Big(\log(P)\cdot\alpha + \frac{n^2}{P^{2/3}}\cdot\beta\Big) \end{aligned}$$

with base-case cost (sequential execution)

$$T_{\text{Tri-Inv}}(n_0,P) = O(\log(P)\cdot\alpha + n_0^2\cdot\beta)$$

the bandwidth cost goes down at each level and we can execute the base-case sequentially when $n_0 = n/P^{1/3}$, with a total cost of

$$T_{\text{Tri-Inv}}(n,P) = O\Big(\log(P)^2\cdot\alpha + \frac{n^2}{P^{2/3}}\cdot\beta\Big)$$

So triangular inversion has *logarithmic depth* while LU has *polynomial depth*, but using inversion within LU naively would raise the LU latency by another log factor

Memory-efficient recursive LU factorization

In the analysis of recursive LU, we assumed

$$T_{\mathrm{MM}}(n, P) = O\big(\log(P)\cdot\alpha + n^2/P^{2/3}\cdot\beta\big)$$

which requires $n^2/P^{2/3}$ memory, $P^{1/3}$ more than minimal

What if we have only cn^2/P memory for some $c \in [1, P^{1/3}]$?

$$T_{\mathrm{MM}}(n, P, c) = O\big(\sqrt{P/c^3}\log(P)\cdot\alpha + n^2/\sqrt{cP}\cdot\beta\big)$$

Q: Does the additional MM latency cost raise the LU latency cost?
A/Q: Naively yes, but could we do something about it?
A: Yes, we could increase c for small subproblems.
What should we set the base case dimension to (previously $n_0 = n/P^{2/3}$)?

$$T_{\mathrm{bc}}(n, n_0) = O\Big((n/n_0)(\log(P)\cdot\alpha + n_0^2\cdot\beta)\Big)$$

$$T_{\mathrm{bc}}\Big(n, \frac{n}{\sqrt{cP}}\Big) = O\Big(\sqrt{cP}\Big(\log(P)\cdot\alpha + \frac{n^2}{cP}\cdot\beta\Big)\Big) = O\Big(\sqrt{cP}\log(P)\cdot\alpha + \frac{n^2}{\sqrt{cP}}\cdot\beta\Big)$$

Short pause

Course projects and homework

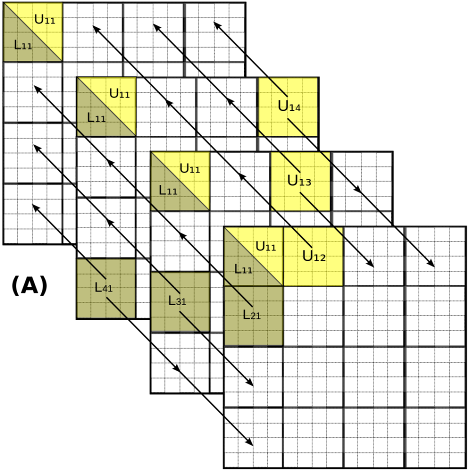
Course projects

- the choice of project will be flexible
- doing something in your current research area is encouraged
- first proposal deadline pushed back a week to Sep 28
- I am happy to give feedback or ideas over email or in person

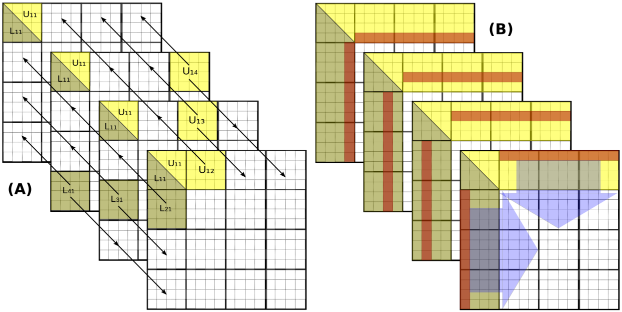
Homework 2

- is due Sep 21
- post questions on Piazza or come to office hours!

2.5D LU factorization



2.5D LU factorization



2.5D LU factorization