CS 598: Communication Cost Analysis of Algorithms
Lecture 16: Tree contraction, Euler tour, list ranking, connectivity, and MST

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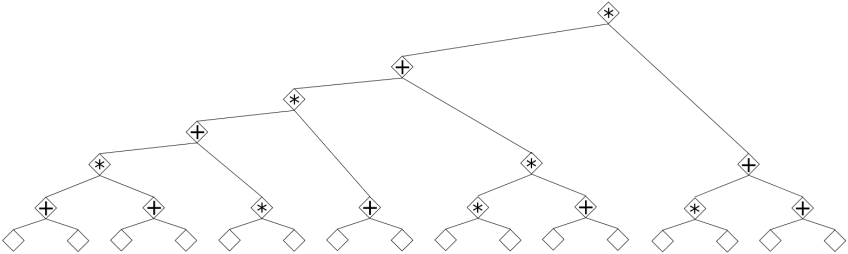
Parallel prefix (scan)

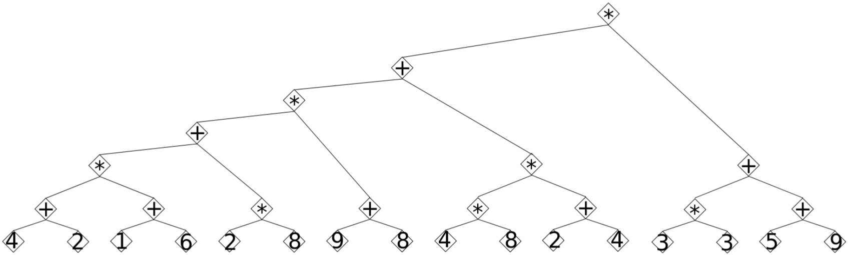
Before continuing with tree contraction, lets consider a simpler problem

- **parallel prefix**: given array $v \in \mathbb{R}^n$, compute $P(v) = w \in \mathbb{R}^n$, so $w(i) = \sum_{j=1}^{i-1} v(j)$
- compute $z = P(y)$ recursively where $y \in \mathbb{R}^{n/2}$ and $y(i) = v(2i) + v(2i+1)$
- then obtain $w(2i) = z(i-1)$, $w(2i+1) = z(i-1) + v(2i)$ where $z(0) = 0$
- can compute with $O(\log(n))$ steps and n processors in PRAM
- Q: how many steps if we use $n/\log_2(n)$ processors?
- A: $O(\log(n))$, for recursive step i need $\max(1, \log_2(n)/2^i)$ steps, total less than $3\log_2(n)$

Tree contraction: expression evaluation ex.



Tree contraction: expression evaluation ex.



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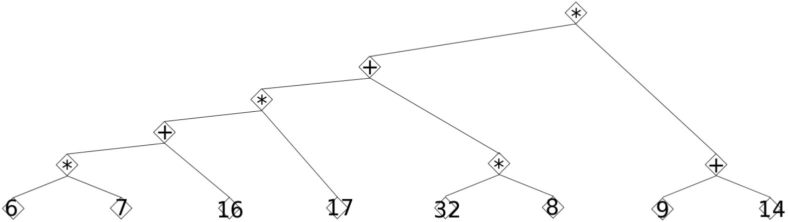

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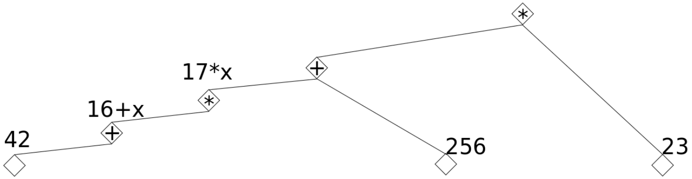
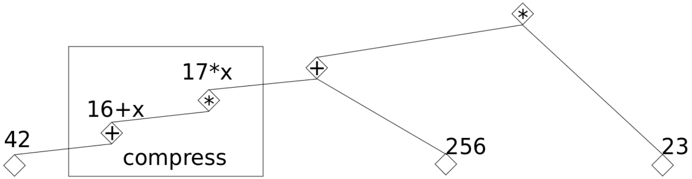


Tree contraction: expression evaluation ex.



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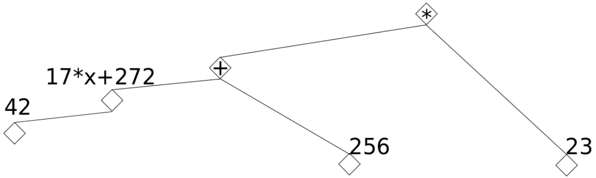

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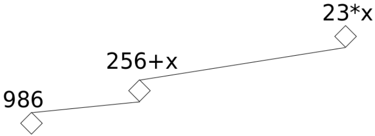
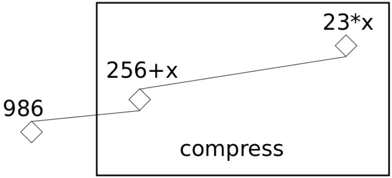
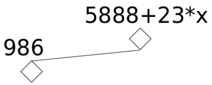


Tree contraction: expression evaluation ex.



Tree contraction: expression evaluation ex.

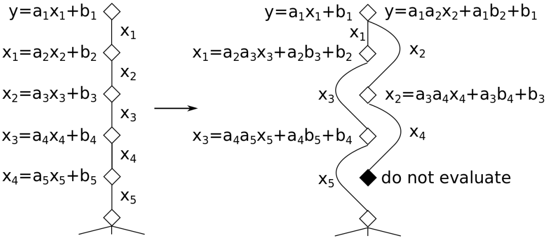


Tree contraction: expression evaluation ex.

Deterministic rake-compress

For a binary tree, raking leaves can be done in $O(1)$ steps

- consider larger branch factors for a boolean expression tree
- Q: if each node computes $\vee$ or $\wedge$, how can we rake in 1 CRCW PRAM step?
- A: if $\vee$, write 1 for all children marked 1, if $\wedge$, write 0 for all children marked 0 (any conflict resolution is correct)
- rake can be done deterministically, by splitting each chain

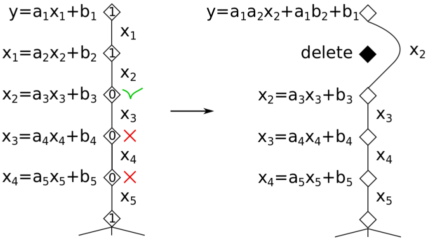


- worst case: chain of length n, completes in $O(\log(n))$ steps but $\Theta(n)$ nodes require work at each step

Randomized parallel compress

Randomization enables a compress step that actually removes nodes

- randomly assign 1 or 0 to each node in the chain
- pointer chase from every node marked 0 whose parent is marked 1



- each rake-compress step decreases the number of nodes by 7/8 w.h.p.
- Q: why does this give us an algorithm that requires $O(\log(n))$ steps and only $O(n/\log(n))$ processors?
- A: first rake-compress with $O(n/\log(n))$ processors takes $O(\log(n))$ steps, each subsequent rake-compress requires a factor of 7/8 fewer steps, so total $O(\log(n))$

Randomized Miller and Reif algorithm in BSP

So, how do we do tree contraction in the BSP model?

- perform $O(n)$ accesses and pointer chases needed in a PRAM step using $O(1)$ BSP supersteps and $O(n/P)$ communication
- with each step of rake-compress we decrease the number of nodes and accesses geometrically
- need to assume the nodes/accesses are load balanced (can randomly permute initially)
- the communication cost then goes down geometrically
- after $O(\log(P))$ steps, the size of the tree is $O(n/P)$, so we can collect all nodes on one processor and contract the tree locally
- the total cost is then

$$O(n/P \cdot \beta + \log(P) \cdot \alpha)$$

Indexing elements of a linked list

List ranking is closely related to tree contraction

- given a linked list p of size n, compute the distance $d_p(i)$ from the end of the list for each element
- more generally, scan on a linked list
    - trivial linear time algorithm sequentially
    - can convert to array via first definition, do scan on array, convert back
    - can also perform scan by contracting list and expanding back
- compress by pointer jumping: e.g. compute $q(i) = p(p(i))$, compute $d_q(i)$ then $d_p(i) = 1 + 2d_q(i)$, $d_p(p(i)) = 2d_q(i)$
- same problem as in compress for tree contraction: how to resolve conflicts?
- again randomized solution is good, assign 0 or 1 to each i, pointer chase if 0 and $p(i)$ is 1
    - to get $d_p(i)$ keep track of non-unit neighbor distances while recursing
- same asymptotic cost as rake-compress, easy in EREW

m-bridges

Efficient computation of tree contraction in EREW can be done by decomposing into n/P-bridges[1]

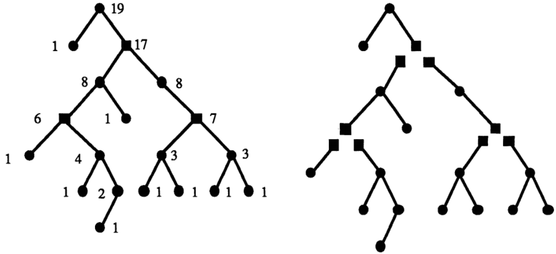


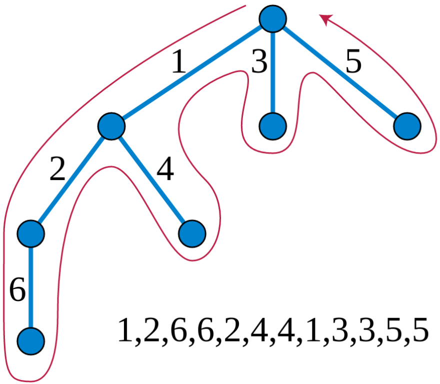
Figure 9.1: The Decomposition of a Tree into its 5-Bridges.

and contracting each bridge to a vertex

[1] diagram source and further information: Gazit, Miller, Teng 1988

Euler tour

We can find the bridges via an **Euler tour**[2]



followed by a list ranking (prefix sum) on the Euler tour tree

[2] diagram source: Wikipedia (David Eppstein)

Cost of tree contraction in other models

Both list ranking and rake-compress are based on pointer-chasing

- how expensive is it to perform n chases with P processors?
  - in PRAM, n/P steps
  - in BSP, 1 superstep, $O(n/P)$ communication
  - in the ideal cache model, with cache line of size L, $O(Ln/P)$ memory bandwidth cost (each chase is likely a cache miss)
  - in the $\alpha - \beta$ model, we have all-to-all-v
    - by direct send: n/P communication with $\min(n/P, P-1)$ messages
    - by butterfly all-to-all (if load-balanced) $O(\frac{n}{P}\log(P)\cdot\beta + \log(P)\cdot\alpha)$
    - so list ranking and tree contraction cost a factor of $O(\log(P))$ more than in BSP
    - the amount of work is $O(n/P\cdot\gamma)$, flop-to-byte ratio $O(1/\log(P))$
- conclusion: pointer chases require lots of messages and random (unstructured) memory accesses
- we can do tree contraction faster, if each processor starts with a subtree (even better, n/P-bridge)

Short pause

Finding connected components

Consider finding the connected sets of vertices in a (disconnected) graph

- sequentially, there are many linear-time solutions, including BFS
- in parallel things are much more interesting
- Shiloach and Vishkin (1980) provide an efficient CRCW PRAM algorithm
- given a graph with n vertices and m edges, it uses $n + m$ processors to complete in $O(\log(n))$ steps

Parallel algorithm for connectivity

Start with a tree for each node and compute a tree for each connected component

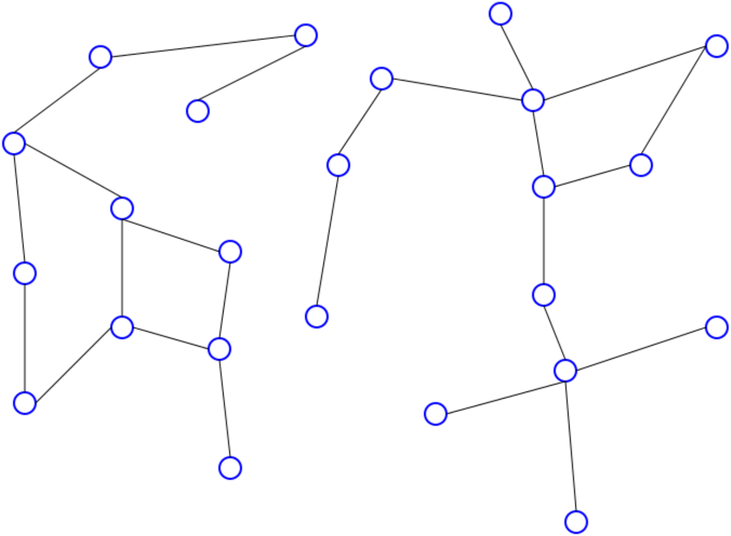
- let each node i store 'parent' $F(i)$
- let a star be any tree of height ≤ 2

The algorithm iterates the following steps

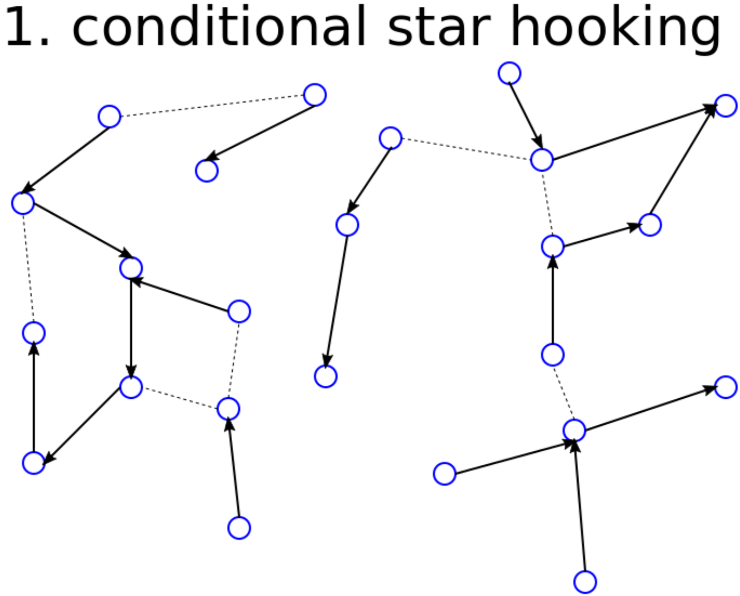
1. *conditional star hooking*: if $(i,j) \in E$, i in star, and $F(i) > F(j)$, perform $F(F(i)) \leftarrow F(j)$ (for every star, some hook may succeed)
2. *unconditional star hooking*: if $(i,j) \in E$, i in star, and $F(i) \neq F(j)$, perform $F(F(i)) \leftarrow F(j)$ (for every star, some hook succeeds)
3. *shortcutting*: (pointer chasing) if i not in star, $F(i) \leftarrow F(F(i))$

and terminates when all nodes are in a star (no hook occurs)

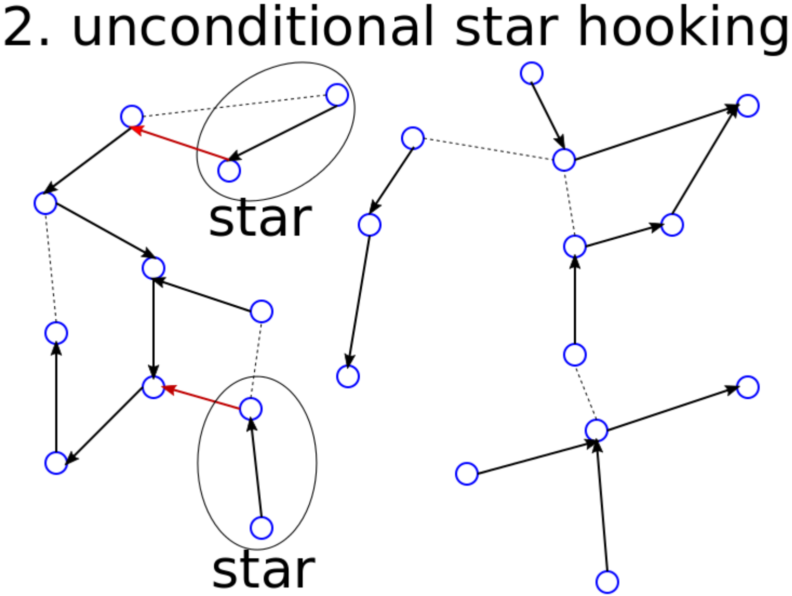
A graph with two connected components



First iteration



First iteration



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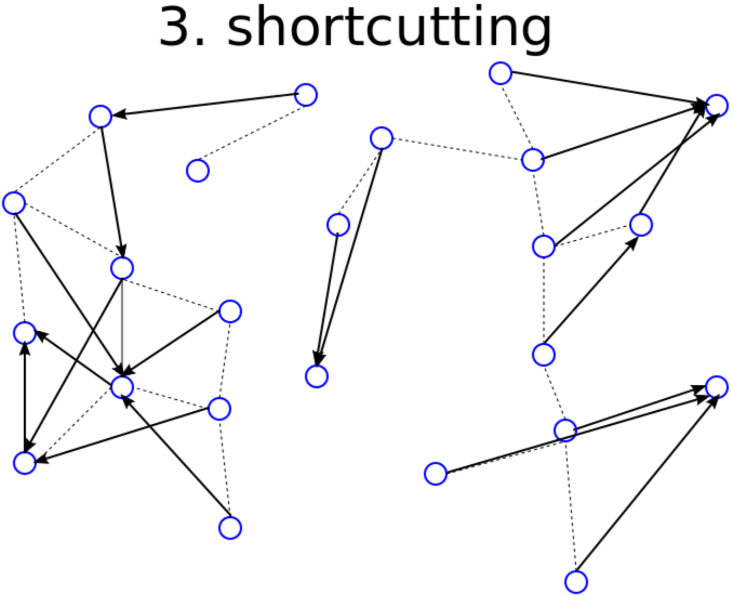
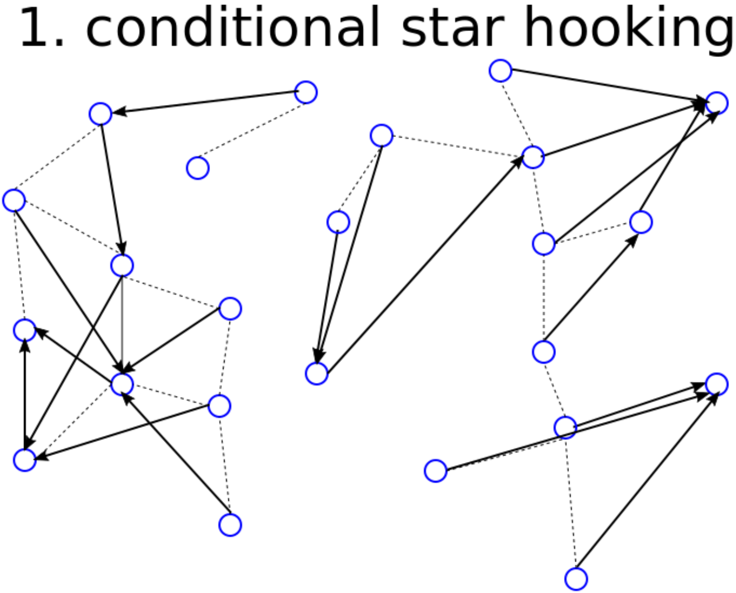
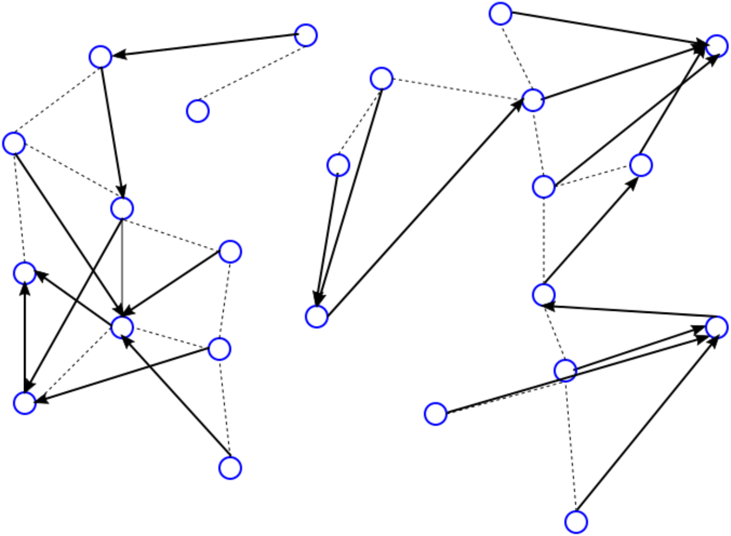


Second iteration



Second iteration

2. unconditional star hooking



Analysis of parallel tree connectivity

Algorithm converges after $O(\log(n))$ iterations

- sum of tree heights (starts at n) decreases by a factor of at least $3/2$ every iteration
  - steps 1 and 2 will hook every star to a tree
  - step 3 will decrease the height of every tree by $3/2$
- requires $O(n + m)$ work per step
- $O(\log(n))$ steps with $O(n + m)$ processors in PRAM
- Q: in BSP, can we do $O(\log(P))$ rather than $O(\log(n))$ steps
- A: not easily, cost proportional to $O(\log(n))$ SpMVs with adjacency matrix, plus pointer chasing

Minimal spanning tree (MST)

Given graph G construct spanning tree with minimal sum of edge weights

- if G not connect, spanning tree forest is desired
- Prim's algorithm: start a tree from random vertex, connect minimal edge to tree
- Kruskal's algorithm: start a tree at every vertex, add minimal edge that connects two trees
  - works for finding forests
  - given two connected parts of the spanning tree (incl. single vertex), minimal edge connecting these must be in the spanning tree
  - this condition suggests a parallel algorithm

Parallel MST algorithm

We follow the approach Shiloach and Vishkin, which is similar to connectivity

- algorithm works for CRCW PRAM with priorities (processor with smallest index wins write conflict)
- start by sorting edges by weight across processors
- perturb each edge weight to make all different or break ties dynamically
- algorithm consists of similar steps
    1. *unconditional star hooking*: if $(i, j) \in E$, i in star, and $F(i) \neq F(j)$, perform $F(F(i)) \leftarrow F(j)$ (for every star, minimal-weight hook succeeds)
    2. *shortcutting*: (pointer chasing) if i not in star, $F(i) \leftarrow F(F(i))$
- if we don't have priorities, need $O(\log(n))$ steps to calculate the minimal-weight hook at every iteration