

Parallel Numerical Algorithms

Chapter 2 – Parallel Thinking

Section 2.1 – Parallel Algorithm Design

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Outline

- 1 Computational Model
- 2 Design Methodology
 - Partitioning
 - Communication
 - Agglomeration
 - Mapping
- 3 Example

Computational Model

- *Task*: a subset of the overall program, with a set of inputs and outputs
- *Parallel computation*: a program that executes two or more tasks concurrently
- *Communication channel*: connection between two tasks over which information is passed (messages are sent and received) periodically
- For now we work with the following messaging semantics
 - *send* is *nonblocking*: sending task resumes execution immediately
 - *receive* is *blocking*: receiving task blocks execution until requested message is available

Example: Laplace Equation in 1-D

- Consider Laplace equation in 1-D

$$u''(t) = 0$$

on interval $a < t < b$ with BC

$$u(a) = \alpha, \quad u(b) = \beta$$

- Seek approximate solution vector \mathbf{u} such that

$$u_i \approx u(t_i) \quad \text{at mesh points} \quad t_i = a + ih, \forall i \in 0, \dots, n+1,$$

where $h = (b - a)/(n + 1)$

Example: Laplace Equation in 1-D

- Finite difference approximation

$$u''(t_i) \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$$

yields tridiagonal system of algebraic equations

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = 0, \quad i = 1, \dots, n,$$

for u_i , $i = 1, \dots, n$, where $u_0 = \alpha$ and $u_{n+1} = \beta$

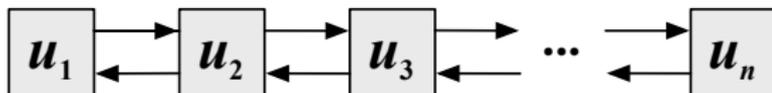
- Starting from initial guess $\mathbf{u}^{(0)}$, compute Jacobi iterates

$$u_i^{(k+1)} = \frac{u_{i-1}^{(k)} + u_{i+1}^{(k)}}{2}, \quad i = 1, \dots, n,$$

for $k = 1, \dots$ until convergence

Example: Laplace Equation in 1-D

- Define n tasks, one for each u_i , $i = 1, \dots, n$
- Task i stores initial value of u_i and updates it at each iteration until convergence
- To update u_i , necessary values of u_{i-1} and u_{i+1} obtained from neighboring tasks $i - 1$ and $i + 1$



- Tasks 1 and n determine u_0 and u_{n+1} from BC

Example: Laplace Equation in 1-D

```
initialize  $u_i$ 
for  $k = 1, \dots$ 
  if  $i > 1$ , send  $u_i$  to task  $i - 1$            { send to left neighbor }
  if  $i < n$ , send  $u_i$  to task  $i + 1$            { send to right neighbor }
  if  $i < n$ , rcv  $u_{i+1}$  from task  $i + 1$        { receive from right neighbor }
  if  $i > 1$ , rcv  $u_{i-1}$  from task  $i - 1$        { receive from left neighbor }
  wait for sends to complete
   $u_i = (u_{i-1} + u_{i+1})/2$                    { update my value }
end
```

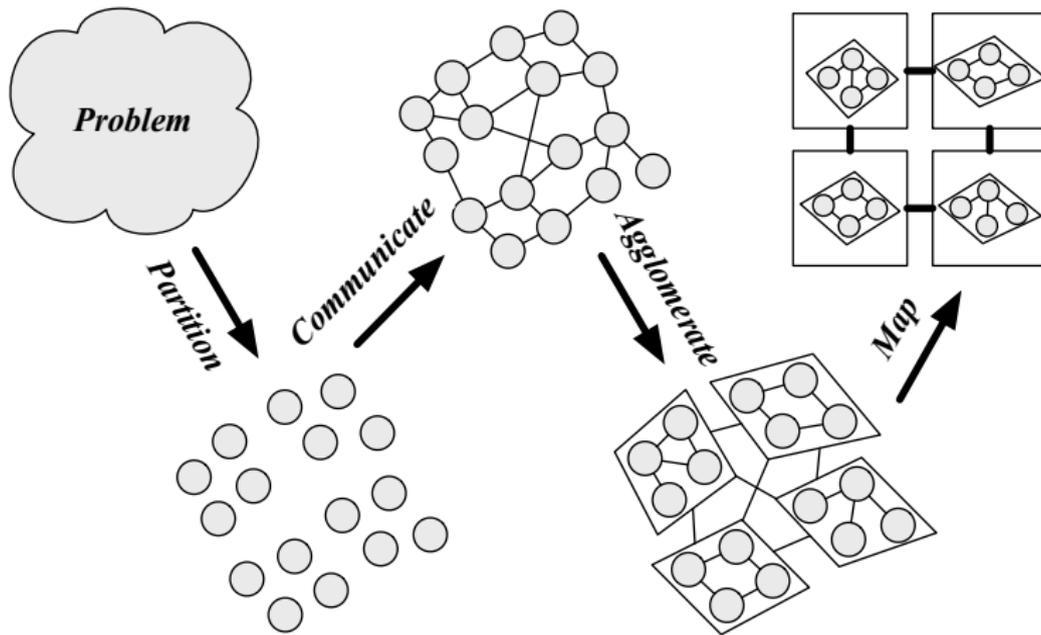
Mapping Tasks to Processors

- Tasks must be assigned to physical processors for execution
- Tasks can be mapped to processors in various ways, including multiple tasks per processor
- Semantics of program should not depend on number of processors or particular mapping of tasks to processors
- Performance usually sensitive to assignment of tasks to processors due to concurrency, workload balance, communication patterns, etc.
- Computational model maps naturally onto distributed-memory multicomputer using message passing

Four-Step Design Methodology

- *Partition*: Decompose problem into fine-grain tasks, maximizing number of tasks that can execute concurrently
- *Communicate*: Determine communication pattern among fine-grain tasks, yielding *task graph* with fine-grain tasks as nodes and communication channels as edges
- *Agglomerate*: Combine groups of fine-grain tasks to form fewer but larger coarse-grain tasks, thereby reducing communication requirements
- *Map*: Assign coarse-grain tasks to processors, subject to tradeoffs between communication costs and concurrency

Four-Step Design Methodology



Graph Embeddings

- Target network may be *virtual network topology*, with nodes usually called *processors* or *processes*
- Overall design methodology is composed of sequence of graph embeddings:
 - fine-grain task graph to coarse-grain task graph
 - coarse-grain task graph to virtual network graph
 - virtual network graph to physical network graph
- Depending on circumstances, one or more of these embeddings may be skipped
- An alternative methodology is to map tasks and communication onto a graph of the network topology laid out in time, similar to the way we defined butterfly protocols

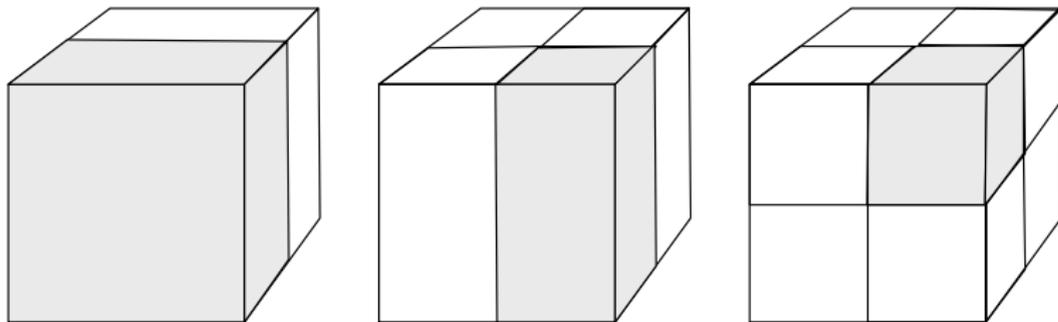
Partitioning Strategies

- *Domain partitioning*: subdivide geometric domain into subdomains
- *Functional decomposition*: subdivide algorithm into multiple logical components
- *Independent tasks*: subdivide computation into tasks that do not depend on each other (*embarrassingly parallel*)
- *Array parallelism*: subdivide data stored in vectors, matrices, or other arrays
- *Divide-and-conquer*: subdivide problem recursively into tree-like hierarchy of subproblems
- *Pipelining*: subdivide sequences of tasks performed by the algorithm on each piece of data

Desirable Properties of Partitioning

- Maximum possible *concurrency* in executing resulting tasks, ideally enough to keep all processors busy
- Number of tasks, rather than size of each task, grows as overall problem size increases
- Tasks reasonably uniform in size
- Redundant computation or storage avoided

Example: Domain Decomposition



3-D domain partitioned along one (left), two (center), or all three (right) of its dimensions

With 1-D or 2-D partitioning, minimum task size grows with problem size, but not with 3-D partitioning

Communication Patterns

- Communication pattern determined by data dependences among tasks: because storage is local to each task, any data stored or produced by one task and needed by another must be communicated between them
- Communication pattern may be
 - local or global
 - structured or random
 - persistent or dynamically changing
 - synchronous or sporadic

Desirable Properties of Communication

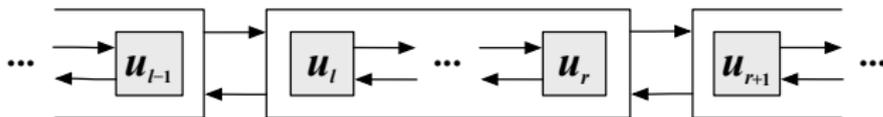
- Frequency and volume minimized
- Highly localized (between neighboring tasks)
- Reasonably uniform across channels
- Network resources used concurrently
- Does not inhibit concurrency of tasks
- Overlapped with computation as much as possible

Agglomeration

- Increasing task sizes can reduce communication but also potentially reduces concurrency
- Subtasks that can't be executed concurrently anyway are obvious candidates for combining into single task
- Maintaining balanced workload still important
- Replicating computation can eliminate communication and is advantageous if result is cheaper to compute than to communicate

Example: Laplace Equation in 1-D

- Combine groups of consecutive mesh points t_i and corresponding solution values u_i into coarse-grain tasks, yielding p tasks, each with n/p of u_i values



- Communication is greatly reduced, but u_i values within each coarse-grain task must be updated sequentially

Example: Laplace Equation in 1-D

```
initialize  $u_l, \dots, u_r$ 
for  $k = 1, \dots$ 
  if  $j > 1$ , send  $u_l$  to task  $j - 1$            { send to left neighbor }
  if  $j < p$ , send  $u_r$  to task  $j + 1$          { send to right neighbor }
  if  $j < p$ , recv  $u_{r+1}$  from task  $j + 1$      { receive from right neighbor }
  if  $j > 1$ , recv  $u_{l-1}$  from task  $j - 1$      { receive from left neighbor }
  for  $i = l$  to  $r$ 
     $\bar{u}_i = (u_{i-1} + u_{i+1})/2$              { update local values }
  end
  wait for sends to complete
   $u = \bar{u}$ 
end
```

Overlapping Communication and Computation

- Updating of solution values u_i is done only after all communication has been completed, but only two of those values actually depend on awaited data
- Since communication is often *much* slower than computation, initiate communication by sending all messages first, then update all “interior” values while awaiting values from neighboring tasks
- Much (possibly *all*) of updating can be done while task would otherwise be idle awaiting messages
- Performance can often be enhanced by overlapping communication and computation in this manner

Example: Laplace Equation in 1-D

```

initialize  $u_l, \dots, u_r$ 
for  $k = 1, \dots$ 
    if  $j > 1$ , send  $u_l$  to task  $j - 1$            { send to left neighbor }
    if  $j < p$ , send  $u_r$  to task  $j + 1$          { send to right neighbor }
    for  $i = l + 1$  to  $r - 1$ 
         $\bar{u}_i = (u_{i-1} + u_{i+1})/2$            { update local values }
    end
    if  $j < p$ , recv  $u_{r+1}$  from task  $j + 1$    { receive from right neighbor }
     $\bar{u}_r = (u_{r-1} + u_{r+1})/2$            { update local value }
    if  $j > 1$ , recv  $u_{l-1}$  from task  $j - 1$    { receive from left neighbor }
     $\bar{u}_l = (u_{l-1} + u_{l+1})/2$            { update local value }
    wait for sends to complete
     $u = \bar{u}$ 
end
    
```

Surface-to-Volume Effect

- For domain decomposition,
 - *computation* is proportional to *volume* of subdomain
 - *communication* is (roughly) proportional to *surface area* of subdomain
- Higher-dimensional decompositions have more favorable surface-to-volume ratio
- Partitioning across more dimensions yields more neighboring subdomains but smaller total volume of communication than partitioning across fewer dimensions

Mapping

- As with agglomeration, mapping of coarse-grain tasks to processors should maximize concurrency, minimize communication, maintain good workload balance, etc
- But connectivity of coarse-grain task graph is inherited from that of fine-grain task graph, whereas connectivity of target interconnection network is independent of problem
- Communication channels between tasks may or may not correspond to physical connections in underlying interconnection network between processors

Mapping

- Two communicating tasks can be assigned to
 - one processor, avoiding interprocessor communication but sacrificing concurrency
 - two adjacent processors, so communication between the tasks is directly supported, or
 - two nonadjacent processors, so message routing is required
- In general, finding optimal solution to these tradeoffs is NP-complete, so heuristics are used to find effective compromise

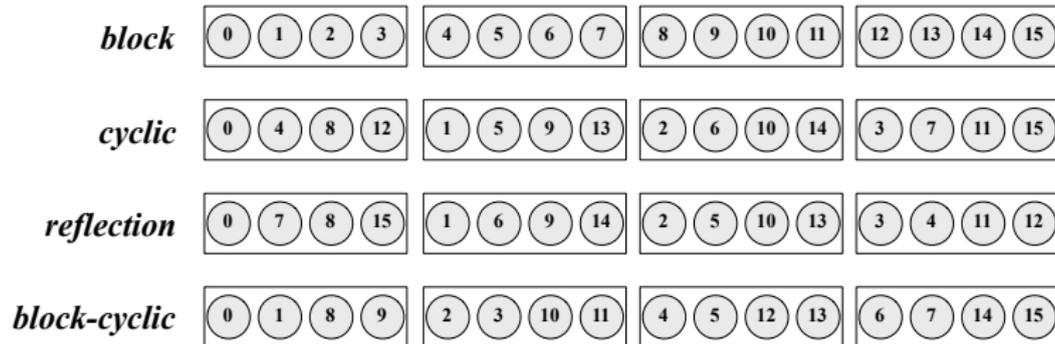
Mapping

- For many problems, task graph has regular structure that can make mapping easier
- If communication is mainly global, then communication performance may not be sensitive to placement of tasks on processors, so random mapping may be as good as any
- Random mappings sometimes used deliberately to avoid communication *hot spots*, where some communication links are oversubscribed with message traffic

Mapping Strategies

- With n tasks and p processors consecutively numbered in some ordering,
 - *block mapping*: blocks of n/p consecutive tasks are assigned to successive processors
 - *cyclic mapping*: task i is assigned to processor $i \bmod p$
 - *reflection mapping*: like cyclic mapping except tasks are assigned in reverse order on alternate passes
 - *block-cyclic mapping* and *block-reflection mapping*: blocks of tasks assigned to processors as in cyclic or reflection
- For higher-dimensional grid, these mappings can be applied in each dimension

Examples of Mappings



Dynamic Mapping

- If task sizes vary *during* computation or can't be predicted in advance, tasks may need to be reassigned to processors dynamically to maintain reasonable workload balance throughout computation
- To be beneficial, gain in load balance must more than offset cost of communication required to move tasks and their data between processors
- Dynamic load balancing usually based on local exchanges of workload information (and tasks, if necessary), so work diffuses over time to be reasonably uniform across processors

Task Scheduling

- With multiple tasks per processor, execution of those tasks must be scheduled over time
- For shared-memory, any idle processor can simply select next ready task from common pool of tasks, or use *work stealing* by taking a task assigned to the task-queue of another processor
- For distributed-memory, the manager/worker paradigm can implement a common task pool, with manager dispatching tasks to workers
- Better scalability is achieved dynamic load balancing strategies in distributed-memory by periodic global or hierarchical rebalancing

Task Scheduling

- For completely decentralized scheme, it can be difficult to determine when overall computation has been completed, so termination detection scheme is required
- With multithreading, task scheduling can conveniently be driven by availability of data: whenever executing task becomes idle awaiting data, another task is executed
- For problems with regular structure, it is often possible to determine mapping in advance that yields reasonable load balance and natural order of execution

Example: Atmospheric Flow Model

- Fluid dynamics of atmosphere modeled by system of partial differential equations
- 3-D problem domain discretized by $n_x \times n_y \times n_z$ mesh of points
- Vertical dimension (altitude) z , much smaller than horizontal dimensions (latitude and longitude) x and y , so $n_z \ll n_x, n_y$
- Derivatives in PDEs approximated by finite differences
- Simulation proceeds through successive discrete steps in time

Example: Atmospheric Flow Model

Partition:

- Each fine-grain task computes and stores data values (pressure, temperature, etc) for one mesh point
- Large-scale problems may require millions or billions of mesh points / tasks

Communicate:

- Finite difference computations at each mesh point use 9-point horizontal stencil and 3-point vertical stencil
- Solar radiation computations require communication throughout each vertical column of mesh points
- Global communication to compute total mass of air over domain

Example: Atmospheric Flow Model

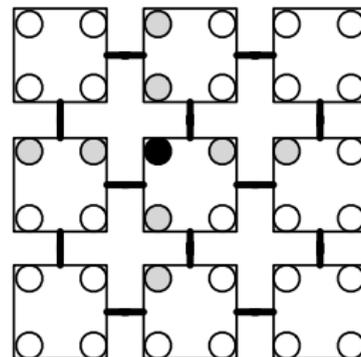
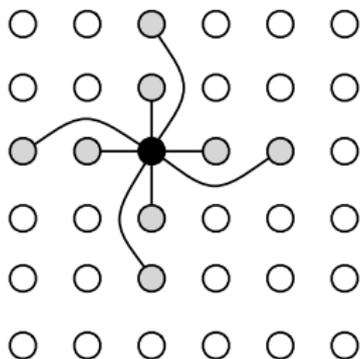
Agglomerate:

- Combine horizontal mesh points in $b \times b$ blocks into coarse-grain tasks to reduce communication for finite differences to exchanges between adjacent nodes
- Combine each vertical column of mesh points into single task to eliminate communication for solar computations
- Yields $n_x/b \times n_y/b$ coarse-grain tasks

Map:

- Cyclic or random mapping reduces load imbalance due to solar computations

Example: Atmospheric Flow Model



Horizontal finite difference stencil for typical point (shaded black) in mesh for atmospheric flow model before (left) and after (right) agglomeration with $b = 2$

References

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