Parallel Numerical Algorithms
Chapter 5 – Eigenvalue Problems
Section 5.1 – QR Factorization

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CS 554 / CSE 512
Outline

1. QR Factorization
2. Householder Transformations
3. Givens Rotations
For given $m \times n$ matrix $A$, with $m > n$, *QR factorization* has form

$$A = Q \begin{bmatrix} R \\ O \end{bmatrix}$$

where matrix $Q$ is $m \times m$ and orthogonal, and $R$ is $n \times n$ and upper triangular.

Can be used to solve linear systems, least squares problems, and eigenvalue problems.

As with Gaussian elimination, zeros are introduced successively into matrix $A$, eventually reaching upper triangular form, but using orthogonal transformations instead of elementary eliminators.
Methods for QR Factorization

- Householder transformations (elementary reflectors)
- Givens transformations (plane rotations)
- Gram-Schmidt orthogonalization
Householder Transformations

- **Householder transformation** has form

\[
H = I - 2\frac{vv^T}{v^Tv}
\]

where \(v\) is nonzero vector.

- From definition, \(H = H^T = H^{-1}\), so \(H\) is both orthogonal and symmetric.

- For given vector \(a\), choose \(v\) so that

\[
Ha = \begin{bmatrix}
\alpha \\
0 \\
\vdots \\
0
\end{bmatrix} = \alpha \begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix} = \alpha e_1
\]
Substituting into formula for $H$, we see that we can take

$$v = a - \alpha e_1$$

and to preserve norm we must have $\alpha = \pm \|a\|_2$, with sign chosen to avoid cancellation.
Householder QR Factorization

\[ \text{for } k = 1 \text{ to } n \]
\[ \alpha_k = -\text{sign}(a_{kk}) \sqrt{a_{kk}^2 + \cdots + a_{mk}^2} \]
\[ v_k = \begin{bmatrix} 0 & \cdots & 0 & a_{kk} & \cdots & a_{mk} \end{bmatrix}^T - \alpha_k e_k \]
\[ \beta_k = v_k^T v_k \]
\[ \text{if } \beta_k = 0 \text{ then} \]
\[ \text{continue with next } k \]
\[ \text{for } j = k \text{ to } n \]
\[ \gamma_j = v_k^T a_j \]
\[ a_j = a_j - (2\gamma_j / \beta_k) v_k \]
\[ \text{end} \]
\[ \text{end} \]
A Householder matrix $H$ is represented by $H = I - uu^T$, i.e. a rank-1 perturbation of the identity.

We can combine $r$ Householder matrices $H_1, \ldots, H_r$ into a rank-$r$ perturbation of the identity

$$\tilde{H} = \prod_{i=1}^{r} H_r = I - UV^T,$$

where $U, V \in \mathbb{R}^{n \times k}$.

Often, $V = UT$ where $T$ is upper-triangular and $U$ is lower-triangular, yielding

$$\tilde{H} = I - UT^T U^T$$

If $H_i = I - u_i u_i^T$, then the $i$th column of $U$ is $u_i$, while $T$ is defined by $T^{-1} + T^{-T} = U^T U$. 

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Parallel Householder QR

- A basis kernel representation of Householder transformations, allows us to update a trailing matrix $B$ as

$$\tilde{H}B = (I - UT^TU^T)B = B - U(T^T(U^TB))$$

with cost $O(n^2r)$

- Performing such updates is essentially as hard as Schur complement updates in LU

- Forming Householder vector $v_k$ is also analogous to computing multipliers in Gaussian elimination

- Thus, parallel implementation is similar to parallel LU, but with Householder vectors broadcast horizontally instead of multipliers
Finding Householder vector $u_i$ requires computation of the norm of the leading vector of the $i$th trailing matrix, creating a latency bottleneck much like that of pivot row selection in partial pivoting.

Other methods need $S = \Theta(\log(p))$ rather than $\Theta(n)$ msgs.

For example, Cholesky-QR and Cholesky-QR2 perform $R = \text{Cholesky}(A^T A)$, $Q = AR^{-1}$ (Cholesky-QR2 does one step of refinement), requiring only a single allreduce, but losing stability.

Unconditional stability and $O(\log(p))$ messages achieved by TSQR algorithm with row-wise recursion (akin to tournament pivoting).

Basis-kernel representation can be recovered by constructing first $r$ columns of $\bar{H}$. 
**Givens Rotations**

- *Givens rotation* operates on pair of rows to introduce single zero

For given 2-vector \( \mathbf{a} = [a_1 \ a_2]^T \), if

\[
  c = \frac{a_1}{\sqrt{a_1^2 + a_2^2}}, \quad s = \frac{a_2}{\sqrt{a_1^2 + a_2^2}}
\]

then

\[
  \mathbf{G}\mathbf{a} = \begin{bmatrix}
    c & s \\
    -s & c
  \end{bmatrix}
  \begin{bmatrix}
    a_1 \\
    a_2
  \end{bmatrix}
  = \begin{bmatrix}
    \alpha \\
    0
  \end{bmatrix}
\]

- Scalars \( c \) and \( s \) are cosine and sine of angle of rotation, and \( c^2 + s^2 = 1 \), so \( \mathbf{G} \) is orthogonal
Givens QR Factorization

- Givens rotations can be systematically applied to successive pairs of rows of matrix $A$ to zero entire strict lower triangle.
- Subdiagonal entries of matrix can be annihilated in various possible orderings (but once introduced, zeros should be preserved).
- Each rotation must be applied to all entries in relevant pair of rows, not just entries determining $c$ and $s$.
- Once upper triangular form is reached, product of rotations, $Q$, is orthogonal, so we have QR factorization of $A$. 
With 1-D partitioning of $A$ by columns, parallel implementation of Givens QR factorization is similar to parallel Householder QR factorization, with cosines and sines broadcast horizontally and each task updating its portion of relevant rows.

With 1-D partitioning of $A$ by rows, broadcast of cosines and sines is unnecessary, but there is no parallelism unless multiple pairs of rows are processed simultaneously.

Fortunately, it is possible to process multiple pairs of rows simultaneously without interfering with each other.
Parallel Givens QR Factorization

- Stage at which each subdiagonal entry can be annihilated is shown here for $8 \times 8$ example

$$
\begin{bmatrix}
\times & \times \\
7 & \times \\
6 & 8 & \times \\
5 & 7 & 9 & \times \\
4 & 6 & 8 & 10 & \times \\
3 & 5 & 7 & 9 & 11 & \times \\
2 & 4 & 6 & 8 & 10 & 12 & \times \\
1 & 3 & 5 & 7 & 9 & 11 & 13 & \times 
\end{bmatrix}
$$

- Maximum parallelism is $n/2$ at stage $n - 1$ for $n \times n$ matrix
Parallel Givens QR Wavefront
Parallel Givens QR Factorization

- Communication cost is high, but can be reduced by having each task initially reduce its entire local set of rows to upper triangular form, which requires no communication.

- Then, in subsequent phase, task pairs cooperate in annihilating additional entries using one row from each of two tasks, exchanging data as necessary.

- Various strategies can be used for combining results of first phase, depending on underlying network topology.

- Parallel partitioning with slanted-panels (slope -2) achieve same scalability as parallel algorithms for LU without pivoting (see [Tiskin 2007]).
Parallel Givens QR Factorization

- With 2-D partitioning of $A$, parallel implementation combines features of 1-D column and 1-D row algorithms.

- In particular, sets of rows can be processed simultaneously to annihilate multiple entries, but updating of rows requires horizontal broadcast of cosines and sines.


References


